for the widest cases and much less for most cases. Its occurrence and sense are such that it does not cause an excess over the error tolerance stated for (12), (13) in comparison therewith.

The usefulness of numerical integration decreases with greater width because u_1 and u_2 approach the pole at $u=1$. This is found to decrease the rate of convergence with smaller intervals.

Acknowledgment

This study has been stimulated by the need for more knowledge of the subject line, which is used where the greatest degree of refinement is afforded. Especially it has been responsive to the opportunity afforded by the "close support" available from such small calculators as the HP-25 and the HP-97. They offer an alternative which has some practical advantages over reading a graph, though they are most valuable for developing the formulas needed in any case. In reference to this and the preceding article, the author is especially grateful to Gerri Fargnoli for the careful typing and to Walter Anderson for his skillful preparation of the graphs.

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Characteristic Impedance of a Rectangular Coaxial Line with Offset Inner Conductor

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***Abstract*—The singular-integral-equation technique is used to derive the capacitance and, hence, characteristic impedance of a rectangular coaxial line with a zero-thickness inner conductor. The position of the inner conductor is arbitrary, but its orientation is assumed to be parallel to the top and bottom walls of the outer conductor. Simple yet very accurate formulas for the capacitance and characteristic impedance are found in terms of complete elliptic integrals.**

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I. Introduction

THE CROSS SECTION of the rectangular transmission line analyzed in this paper is shown in Fig. 1. The zero-thickness inner conductor is arbitrarily situated but is parallel to the x axis. Both conductors are perfectly conducting, and the medium between the two conductors is a homogeneous dielectric.

This type of transmission line has found use in some EMI measurement systems [1] as a transducer for coupling EM energy from the equipment under test (EUT) into the TEM mode of the transmission line. The EUT is usually located between the inner and outer conductors

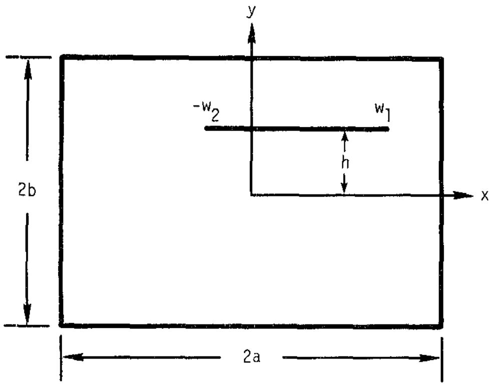

Fig. 1. Cross section of a rectangular coaxial line with an offset inner conductor.

but is isolated from them. In many cases the inner conductor is offset vertically in order to accommodate larger EUT's. The characteristic impedance of the transmission line becomes an important design consideration in order to maximize either the transmitted or received power in susceptibility or radiated emission tests, respectively. Since the transmission line usually connects through a tapered section to an ordinary 50-Ω coaxial line, the nominal characteristic impedance must be chosen to be approximately 50 Ω.

In the past a significant amount of work has been done in analyzing the transmission-line properties of various rectangular coaxial waveguiding structures. The solutions obtained are generally for two classes of rectangular lines, those for which the thickness of the inner conductor is assumed to be either zero or nonzero. Cohn [2], Tippet and Chang [3], and Hachemeister [4] have studied the zero-thickness case, while Anderson [5] and Riblet [6] have obtained results for inner conductors of finite thickness. Chen [7] has summarized results for both cases. In most of these analyses, the inner conductor was assumed to be symmetrically located within the outer conductor. Hachemeister, however, has obtained results for the horizontally offset zero-thickness inner conductor, while Chen has obtained results for the horizontally and vertically offset inner conductor of finite thickness. Chen's result, however, is not applicable in the limiting case when the thickness of the inner conductor reduces to zero. It is the purpose of this paper to investigate the dependence of the characteristic impedance on the various dimensions of the rectangular line. In particular, we will obtain results for a vertically offset inner conductor which has received little attention in the literature. The formulation, however, is also applicable for horizontal offsets.

This problem is formulated using an integral-equation–Green's-function type of formulation. The kernel of the resulting integral equation is split into its singular and nonsingular parts. The nonsingular part is then expanded in terms of Chebyshev polynomials (as suggested in [8]), and the integral equation is inverted using the singular-integral-equation technique [9]. The solution is found in terms of complete elliptic integrals. As will become obvious, this formulation has the advantage of handling the edge condition exactly and eliminates the problems encountered in any numerical solution associated with the discontinuities of the fields near the sharp edges of the inner conductor.

The retention of only a finite number of terms in the expansion for the nonsingular part of the kernel results in an explicit formula for the capacitance. It can be shown, however, that the higher order terms decay as $e^{-m\pi(b-h)/a}$ where m is the summation index. Thus keeping only one term can actually provide a surprisingly accurate result for a wide range of practical geometries (error <0.5 percent for $b/a \geqslant 0.5$, $h=0$, $w_1/w_2=1$). For very large b/a the solution reduces to an exact one which can be obtained using the method of conformal transformation. For finite b/a, the effect of the images of the inner conductor about the top and bottom walls of the outer conductor becomes significant. Our approximation can be thought of as taking into account one or more of these image terms. As long as b/a is not too small, the effect of only a few of the image terms is felt.

II. Formulation

Using Green's theorem, we know that Poisson's equation can be converted into an integral equation for the unknown charge density on the inner conductor as follows:

$$\int_{-w_2}^{w_1} \frac{\rho(x')}{\epsilon} G(x,x')dx' = V, \qquad -w_2 < x < w_1 \tag{1}$$

where $\rho(x')$ is the charge density, ϵ is the dielectric permittivity, and V is the voltage between the inner and outer conductors. $G(x,x')$ represents the Green's function of a rectangular region of cross section $2a \times 2b$ for both observation and source points $\bar{x}$ and $\bar{x}'$, respectively, located on the inner conductor, and is given as

$$G(x,x') = \sum_{m,n} \frac{\psi_{mn}(\bar{x})\psi_{mn}(\bar{x}')}{K_{mn}^2} \tag{2}$$

where

$$\psi_{mn}(\bar{x}) = \frac{1}{\sqrt{ab}} \sin\left[\frac{m\pi}{2a}(x+a)\right] \sin\left[\frac{n\pi}{2b}(h+b)\right]$$

and

$$K_{mn} = \left[\left(\frac{m\pi}{2a}\right)^2 + \left(\frac{n\pi}{2b}\right)^2\right]^{1/2}.$$

The Green's function given in (2) can be written as a sum of two terms, one of which is logarithmically singular for $\bar{x}=\bar{x}'$ and one of which is nonsingular. By extracting the singular part we can convert (1) into a standard form of the singular integral equation which can then be inverted exactly. We can identify the singular part of the Green's function by first noting that we can perform the summation on "n" in (2) exactly. In the remaining summation we replace the coefficients in front of the trigonometric functions by their asymptotic form for large "m." This sum represents the singular part of the Green's function and is given as

$$G_s = \sum_{m=1}^{\infty} \frac{\sin m\theta \sin m\phi}{m\pi} = \frac{1}{4\pi} \ln \left| \frac{1-\cos(\theta+\phi)}{1-\cos(\theta-\phi)} \right| \tag{3}$$

where

$$\theta \equiv \frac{\pi}{2a}(x+a)$$

and

$$\phi \equiv \frac{\pi}{2a}(x'+a).$$

The remaining correction series is nonsingular and is given as

$$G_n = \sum_{m=1}^{\infty} \frac{A_m \sin m\theta \sin m\phi}{m\pi} \tag{4}$$

where

$$A_m = \left[\frac{\cosh\left(\frac{m\pi b}{a}\right) - \cosh\left(\frac{m\pi h}{a}\right)}{\sinh\left(\frac{m\pi b}{a}\right)} - 1 \right]. \tag{5}$$

If we now transform (1) into an equation in terms of θ and ϕ and differentiate with respect to θ, we obtain

$$P\int_{\phi_1}^{\phi_2} f(\phi)\left[\partial_\theta G_s + \partial_\theta G_n\right] d\phi = 0 \tag{6}$$

where

$$\phi_1 = \frac{\pi}{2a}(a-w_2)$$

$$\phi_2 = \frac{\pi}{2a}(a+w_1)$$

$$f(\phi) = \rho(x')$$

and P denotes that the integral is to be interpreted in the principal value sense. $\partial_\theta G_s$ and $\partial_\theta G_n$ from (3) and (4) are given as

$$\partial_\theta G_s = \frac{1}{2\pi} \frac{\sin\phi}{(\cos\theta - \cos\phi)}$$

and

$$\partial_\theta G_n = \frac{1}{\pi} \sum_{m=1}^{\infty} A_m \cos m\theta \sin m\phi \tag{7}$$

so that (6) can be written as follows:

$$\frac{1}{2\pi} P\int_{\phi_1}^{\phi_2} \frac{f(\phi)\sin\phi}{(\cos\theta - \cos\phi)} d\phi = -\int_{\phi_1}^{\phi_2} f(\phi)\partial_\theta G_n d\phi. \tag{8}$$

If we now invoke Schwinger's transformation [10]:

$$\cos\theta = \alpha - \beta v$$

and

$$\cos\phi = \alpha - \beta u$$

with α and β given by

$$\alpha = \frac{1}{2}\left[\cos\phi_1 + \cos\phi_2\right]$$

and

$$\beta = \frac{1}{2}\left[\cos\phi_1 - \cos\phi_2\right]$$

we can transform (8) into the canonical form of the singular integral equation:

$$T_v[F(u)] = H(v) \tag{9}$$

where

$$T_v[F(u)] = \frac{1}{\pi} P\int_{-1}^{1} \frac{F(u)}{u-v} du$$

with

$$F(u) = f(\phi)$$

and

$$H(v) = -2\int_{\phi_1}^{\phi_2} f(\phi)\partial_\theta G_n d\phi. \tag{10}$$

III. Solution

Following the rationale given in [8], it is convenient to expand $H(v)$ given in (10) in terms of Chebyshev polynomials of the second kind U_m as follows:

$$H(v) = \sum_{m=0}^{\infty} C_{m+1} U_m(v) \tag{11}$$

where the C_m's are coefficients to be determined. The inversion of (9) is exactly given as [9, p. 201]

$$F(v) = \frac{1}{\left[1-v^2\right]^{1/2}} \left\{ C_0 - T_v\left[\left[1-u^2\right]^{1/2} H(u)\right]\right\} \tag{12}$$

with C_0 to be determined, so that upon inserting (11) into (12) and using the following identity [11]:

$$T_v\left[\left[1-u^2\right]^{1/2} U_m(u)\right] = -T_{m+1}(v)$$

where T_m is the Chebyshev polynomial of the first kind, one obtains for $F(v)$ the following:

$$F(v) = \frac{1}{\left[1-v^2\right]^{1/2}} \sum_{m=0}^{\infty} C_m T_m(v). \tag{13}$$

Equation (13) is an expression for the charge density on the inner conductor. This quantity must be integrated over the strip to obtain the total charge and thus the capacitance. From its definition, the total charge Q and the capacitance are related as follows:

$$Q = CV = \int_{-w_2}^{w_1} \rho(x')dx' = \frac{2a}{\pi}\int_{\phi_1}^{\phi_2} f(\phi)d\phi$$

$$= \frac{2a\beta}{\pi}\int_{-1}^{1} \frac{F(u)}{\sin\phi} du. \tag{14}$$

Thus, inserting (13) into (14) and expressing $\sin\phi$ in terms of u, one obtains the following:

$$C = \frac{2a}{\pi V} \sum_{m=0}^{\infty} C_m I_m \tag{15}$$

where

$$I_m \equiv \int_{-1}^{1} \frac{T_m(u)\,du}{\left[\left(\frac{1+\alpha}{\beta}-u\right)(1-u)(u+1)\left(u+\frac{1-\alpha}{\beta}\right)\right]^{1/2}}. \tag{16}$$

As shown in Appendix A, I_m can be calculated in terms of complete elliptic integrals [12] of the following modulus:

$$k=\frac{2\sqrt{\beta}}{\left[(1+\beta)^2-\alpha^2\right]^{1/2}}. \tag{17}$$

To complete the solution it remains to calculate the constants C_m in (15). This is done by substituting the expressions for H and F given, respectively, in (11) and (13) into the defining equation (10) and the initial undifferentiated integral equation (1). The equations that result are derived in Appendix B and are given as follows:

$$\sum_{n=0}^{\infty} C_n J_n + 2\sum_{n=0}^{\infty}\sum_{k=0}^{n} p_{kn}\frac{C_k}{\Delta_k}A_{n+1}\frac{\cos\left(\frac{n\pi}{2}\right)}{(n+1)} = \frac{2\pi\epsilon V}{a\beta} \tag{18}$$

and

$$C_{n+1}+\beta\sum_{m=n}^{\infty}\sum_{k=0}^{m-1}(1-\delta_{m0})A_m q_{nm}\frac{C_k}{\Delta_k}p_{k,m-1}=0, \qquad n=0,1,2,\cdots \tag{19}$$

where δ_{m0} is the Kronecker delta, Δ_k is the Neumann factor, and p_{mn} and q_{mn} as defined in Appendix B are, respectively, the expansion coefficients of $\sin(n+1)\phi/\sin\phi$ and $\cos n\phi$ in terms of Chebyshev polynomials $T_m(u)$ and $U_m(u)$ with $u=(\alpha-\cos\phi)/\beta$. A_n is the expansion coefficient of the nonsingular G_n given in (5), and J_n is a canonical integral defined as

$$J_n \equiv \frac{1}{\pi}\int_{\phi_1}^{\phi_2} \frac{T_n\left(\frac{\alpha-\cos\phi}{\beta}\right)\ln\left|\frac{1+\sin\phi}{1-\sin\phi}\right|}{\left[(\cos\phi_1-\cos\phi)(\cos\phi-\cos\phi_2)\right]^{1/2}}\,d\phi. \tag{20}$$

As shown in Appendix A, this integral also can be calculated in terms of the complete elliptic integrals, however, of the modulus complementary to k defined as

$$k'=\left[1-k^2\right]^{1/2}=\left[\frac{(1-\beta)^2-\alpha^2}{(1+\beta)^2-\alpha^2}\right]^{1/2}. \tag{21}$$

In order to solve for the constants C_m, we must invert the infinite set of equations given in (18) and (19). By examining (5) we see that for large m, A_m is given approximately by

$$A_m \sim -e^{-m\pi(b-h)/a}.$$

This exponential convergence allows us to truncate the infinite matrix to one of very small order. The order of the resulting matrix depends of course on the magnitude of $(b-h)/a$. If $(b-h)/a$ is very large, then clearly all of the A_m's are identically zero, and the capacitance is given as

$$C=\frac{2a}{\pi V}C_0 I_0=\frac{4\epsilon I_0}{\beta J_0}=2\epsilon\frac{K(k)}{K(k')} \tag{22}$$

where k was defined in (17). Furthermore, if the center conductor is symmetrically located left and right, then $\alpha=0$ and (22) can be further simplified by applying Gauss' transformation [13]

$$K\left[\frac{2\sqrt{\beta}}{1+\beta}\right]=(1+\beta)K(\beta)$$

so that (22) reduces to

$$C=4\epsilon\frac{K(\beta)}{K(\beta')}. \tag{23}$$

Equations (22) and (23) are the exact solutions for the limiting case of $b/a\to\infty$[4] for nonzero and zero offset, respectively.

We can now obtain first-order corrections by assuming that only A_1 in (18) and (19) is significant, i.e., we let $A_m=0$ for $m>1$. The infinite set of equations then reduces to just three equations in three unknowns which can be solved to give

$$C=\frac{4\epsilon}{\beta}\left[\frac{I_0-2\alpha\beta A_1 I_1+\beta^2 A_1 I_2}{J_0-2\alpha\beta A_1 J_1+\beta^2 A_1 J_2+4A_1}\right].$$

The canonical integrals I_n and J_n are evaluated in Appendix A. Inserting these results into the above equation we then obtain

$$C=2\epsilon\left\{\frac{\left[1+(1-\alpha^2)A_1\right]K-A_1\left[(1+\beta)^2-\alpha^2\right]E}{\left[1-\beta(2+\beta)A_1\right]K'+A_1\left[(1+\beta)^2-\alpha^2\right]E'}\right\}, \qquad \mathrm{mod}\,k. \tag{24}$$

It is interesting to note that while the integrals I_1, I_2, J_1, and J_2 by themselves all are functions of the elliptic integral of the third kind Π, the final result for the capacitance does not contain the Π function. This is an advantage in the numerical evaluation of (24) as both K and E can be computed accurately and efficiently using an algorithm known as the arithmetical-geometric mean (AGM) [14].

IV. Numerical Results

In order to check the accuracy of (24), we will assume that the inner conductor is symmetrically located both left and right, i.e., $\alpha=0$, and up and down, i.e., $h=0$. Equation (24) can then be reduced to the following form using Gauss' transformations [13]:

$$C=4\epsilon\left\{\frac{K\left[1+A_1(2-\beta^2)\right]-2A_1E}{K'\left(1-\beta^2A_1\right)+2A_1E'}\right\}, \qquad \mathrm{mod}\,\beta \tag{25}$$

where

$$A_1=\tanh\left(\frac{\pi b}{2a}\right)-1$$

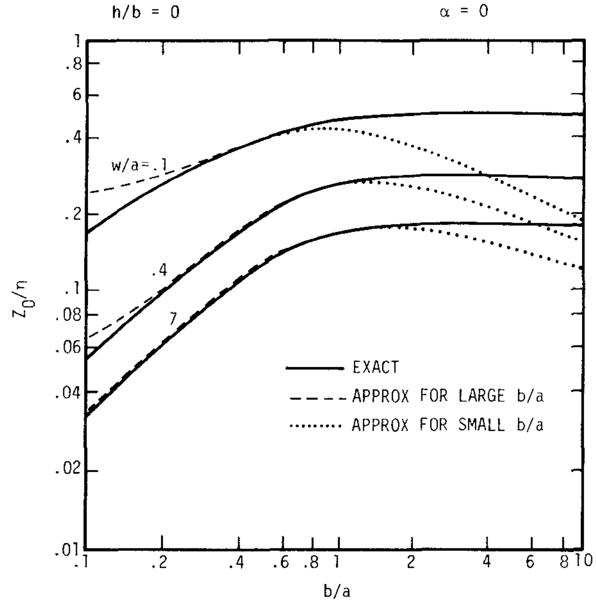

Fig. 2. Comparison of characteristic impedance formulas for a rectangular coaxial line with a symmetrically located inner conductor.

$$\beta=\sin\left(\frac{\pi w}{2a}\right)$$

and

$$w_1=w_2\equiv w.$$

We can now compare this solution to an exact solution for finite b/a which is found using conformal transformation methods (see, e.g., [3]) as

$$C_{ex}=2\epsilon\frac{K(\lambda)}{K(\lambda')} \tag{26}$$

where

$$\lambda'=\sqrt{1-\lambda^2}=\tau'\left(\frac{\operatorname{sn}\zeta}{\operatorname{cn}\zeta}\right)^2$$

$$\zeta=\frac{g}{b}K(\tau')$$

$$g=a-w$$

$$\frac{K(\tau)}{K(\tau')}=\frac{2a}{b}$$

$$\tau'=\sqrt{1-\tau^2}$$

and sn and cn are Jacobian elliptic functions of modulus τ.

In the following curves we will plot the characteristic impedance as a function of various parameters. The characteristic impedance Z_0 is of course obtained from the capacitance as

$$Z_0/\eta=\frac{1}{C/\epsilon}$$

where η is the intrinsic impedance defined as

$$\eta=[\mu/\epsilon]^{1/2}$$

and ϵ and μ are, respectively, the dielectric permittivity and magnetic permeability of the material separating the inner and outer conductors.

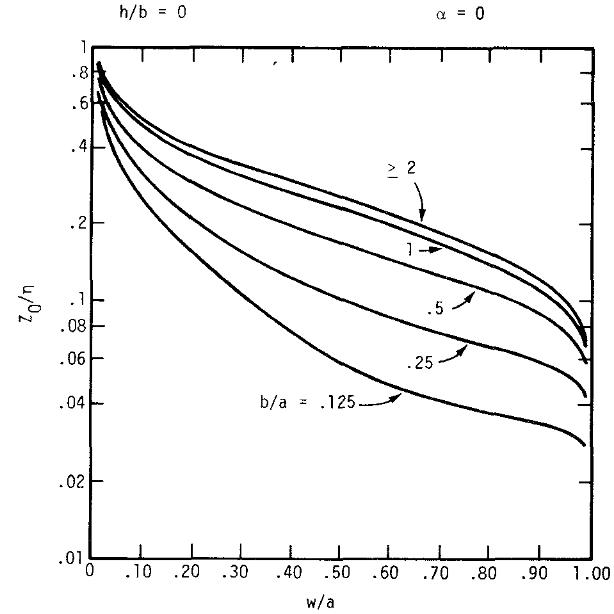

Fig. 3. Design curves for the characteristic impedance based on (25) for a rectangular coaxial line with a symmetrically located inner conductor.

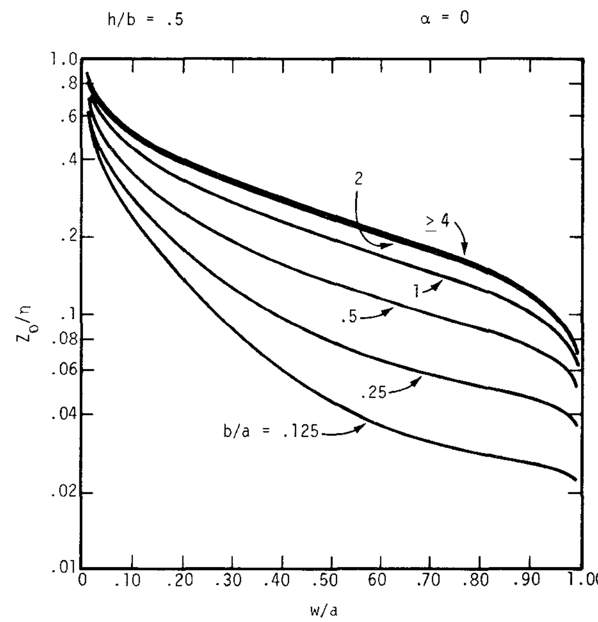

Fig. 4. Design curves for the characteristic impedance based on (25) for a rectangular coaxial line with a vertically offset inner conductor.

Plotted in Fig. 2 are the exact characteristic impedance based on (26), the approximate formula based on (25), and an often used approximation given as [7]

$$Z_0/\eta=\frac{1}{4}\left[\frac{w}{b}+\frac{2}{\pi}\ln\left(1+\coth\frac{\pi a}{2b}\right)\right]^{-1}. \tag{27}$$

In this graph the abscissa is the aspect ratio of the guide b/a, and the three sets of curves correspond to $w/a=0.1$, 0.4, and 0.7. Obviously, the approximate solution given in (25) is good for $b/a\gtrsim 0.35$ while (27) is good for $b/a\lesssim 0.35$. Together the two expressions provide a very accurate result for zero-offset rectangular coaxial lines of arbitrary dimensions.

Plotted in Figs. 3 and 4 are design curves for the characteristic impedance based on (24) but for an arbitrary vertical offset, i.e., with A_1 given by (5). Fig. 3 is for zero offset while Fig. 4 is for an offset ratio h/b of 0.5. The abscissa in these curves is the normalized width of the inner conductor, and the parameter is the aspect ratio of the guide. A computer program for generating curves corresponding to other combinations of offset ratios and aspect ratios can be found in [15]. We note that these curves are quite rapidly varying in the narrow strip ($w/a \ll 1$) and small gap ($g/a \ll 1$) regions. Approximate solutions valid in these regions can be obtained by replacing the elliptic integrals in (24) by their asymptotic forms for modulus either near zero or one. One would then find the familiar logarithmic dependence characteristic of such solutions. These solutions are not only useful, however, in the limiting regions but may also be extended outside of this range as is done, for example, by Wheeler [16].

V. Conclusions

We have presented in this paper a technique for obtaining the capacitance and/or characteristic impedance of a rectangular coaxial line with offset inner conductor. A zero-order solution given in (22) and a first-order solution given in (24) have been obtained; however, the method can be extended to higher orders if one is willing to invert larger matrices. For most practical geometries this is not necessary, as indicated by the accuracy of the approximation in Fig. 2 for $b/a \gtrsim 0.35$. The design curves given in Figs. 3 and 4 enable one to choose appropriate dimensions for the rectangular line to obtain a required characteristic impedance.

Appendix A
Evaluation of the Canonical Integrals

We begin with the integral I_m defined in (16) as

$$I_m \equiv \int_{-1}^{1} \frac{T_m(u)\,du}{\left[\left(\frac{1+\alpha}{\beta}-u\right)(1-u)(u+1)\left(u+\frac{1-\alpha}{\beta}\right)\right]^{1/2}}.$$

This integral can be represented as a linear combination of integrals defined by replacing $T_m(u)$ by u^m. The evaluation of these integrals with u^m in the numerator can be found in [12, p. 113, (254.10)] as

$$\int_{-1}^{1} \frac{u^m du}{\left[\left(\frac{1+\alpha}{\beta}-u\right)(1-u)(u+1)\left(u+\frac{1-\alpha}{\beta}\right)\right]^{1/2}} = (-1)^m \sqrt{\beta}\, k Z_m \quad \text{(A1)}$$

where

$$k = \frac{2\sqrt{\beta}}{\left[(1+\beta^2)-\alpha^2\right]^{1/2}}$$

and

$$Z_m = \left(\frac{1-\alpha}{\beta}\right)^m \sum_{j=0}^{m} \binom{m}{j} \left(\frac{\alpha+\beta-1}{1-\alpha}\right)^j V_j$$

with V_m given recursively as

$$2(m+2)(1-\gamma^2)(k^2-\gamma^2)V_{m+3} = (2m+3)(\gamma^4-2\gamma^2k^2-2\gamma^2+3k^2)V_{m+2} + 2(m+1)(\gamma^2k^2+\gamma^2-3k^2)V_{m+1} + (2m+1)k^2V_m$$

where

$$\gamma^2 = \frac{2\beta}{1+\beta-\alpha}$$

$$V_0 = K(k)$$

$$V_1 = \Pi(\gamma^2, k)$$

$$2(\gamma^2-1)(k^2-\gamma^2)V_2 = (2\gamma^2k^2+2\gamma^2-\gamma^4-3k^2)\cdot\Pi(\gamma^2,k) + \gamma^2E(k) + (k^2-\gamma^2)K(k)$$

and K, E, and Π are complete elliptic integrals of the first, second, and third kinds, respectively.

If the inner conductor is symmetrically located, i.e., $\alpha=0$, then (A1) simplifies to the following form:

$$\beta\int_{-1}^{1} \frac{u^m du}{\left[(1-u^2)(1-\beta^2u^2)\right]^{1/2}} \equiv \beta \bar{A}_m. \quad \text{(A2)}$$

The integral $\bar{A}_m$ is evaluated recursively in [12, p. 191, (310.05)] as

$$\bar{A}_{2m+1} = 0$$

and

$$(2m+1)\beta^2\bar{A}_{2m+2} = 2m(1+\beta^2)\bar{A}_{2m} + (1-2m)\bar{A}_{2m-2}$$

where

$$\bar{A}_0 = 2K(\beta)$$

and

$$\bar{A}_2 = \frac{2}{\beta^2}\left[K(\beta)-E(\beta)\right].$$

The remaining canonical integral J_m is defined in (20) as

$$J_m \equiv \frac{1}{\pi}\int_{\phi_1}^{\phi_2} \frac{T_m\left(\frac{\alpha-\cos\phi}{\beta}\right)\ln\left|\frac{1+\sin\phi}{1-\sin\phi}\right|}{\left[(\cos\phi_1-\cos\phi)(\cos\phi-\cos\phi_2)\right]^{1/2}}\,d\phi.$$

This integral can be evaluated using the method given in [9, pp. 188–192]. In fact, the integral I_r defined in [9, (6.98)] is related to J_m by replacing $T_m((\alpha-\cos\phi)/\beta)$ by $\cos m\phi$. We can thus represent J_m as a linear combination of the I_r's. Using [9, (6.109)] and [9, (6.117)] we find that

$$J_0 = \frac{2}{\left[1-\beta^2\right]^{1/2}}K(k_1) \quad \text{(A3)}$$

and

$$\beta J_2 - 2\alpha J_1 = \frac{2}{\beta}\left[-2+2\left[1-\beta^2\right]^{1/2}E(k_1)\right] - \beta J_0 \quad \text{(A4)}$$

$$k_1=\left[\frac{\bar{\alpha}^2-\bar{\beta}^2}{1-\bar{\beta}^2}\right]^{1/2}$$

and $\bar{\alpha}$ and $\bar{\beta}$ are related to α and β as follows:

$$\bar{\alpha}\bar{\beta}=-\alpha$$

and

$$\bar{\alpha}^2+\bar{\beta}^2=1+\alpha^2-\beta^2, \qquad \bar{\alpha}\geqslant\bar{\beta}.$$

It can be shown that the modulus k' defined in (21) and k_1' the modulus complementary to k_1 defined as

$$k_1'=\left[1-k_1^2\right]^{1/2}$$

satisfy the following relation:

$$k'=\frac{1-k_1'}{1+k_1'}.$$

Thus we can use the following Gauss' transformations [13]

$$K(k_1)=\frac{2}{1+k_1'}K(k')$$

and

$$E(k_1)=(1+k_1')E(k')-k_1'K(k_1)$$

to rewrite (A3) and (A4) in terms of the modulus k' as

$$\beta J_0=2\sqrt{\beta}\;kK(k')$$

and

$$\beta J_2-2\alpha J_1=\frac{4}{\beta}\left[-1+\frac{2\sqrt{\beta}}{k}E(k')-\sqrt{\beta}\;kK(k')\right]-\beta J_0$$

where k was defined in (17).

Appendix B
Derivation of the Matrix Equation

We begin by writing (1) in terms of θ and ϕ as

$$\int_{-w_2}^{w_1}\frac{\rho(x')}{\epsilon}G(x,x')\,dx'=\frac{2a}{\pi}\int_{\phi_1}^{\phi_2}\frac{f(\phi)}{\epsilon}\left[G_s+G_n\right]d\phi=V. \tag{B1}$$

Equation (B1) holds for $\phi_1\leqslant\theta\leqslant\phi_2$, and, without loss of generality, we can set θ to any convenient value within this range in order to make the integrations as simple as possible. We choose $\theta=\pi/2$. This choice places some restrictions on the amount of offset that is allowed. Specifically, the inner conductor must contain the guide center. As shown in [9, p. 178], however, this restriction is a temporary one, and the final result obtained will hold for arbitrary offset.

Upon substituting for $f(\phi)$ from (13) and setting $\theta=\pi/2$, one can write the singular part of (B1) as

$$\frac{2a}{\pi}\int_{\phi_1}^{\phi_2}\frac{f(\phi)}{\epsilon}G_s d\phi=\frac{a\beta}{2\pi\epsilon}\sum_{m=0}^{\infty}C_mJ_m$$

where

$$J_m\equiv\frac{1}{\pi}\int_{\phi_1}^{\phi_2}\frac{T_m\left(\frac{\alpha-\cos\phi}{\beta}\right)\ln\left|\frac{1+\sin\phi}{1-\sin\phi}\right|}{\left[(\cos\phi_1-\cos\phi)(\cos\phi-\cos\phi_2)\right]^{1/2}}d\phi.$$

As shown in Appendix A, J_m can be evaluated in terms of complete elliptic integrals. The nonsingular part of (B1) is evaluated by transforming the integration into an equation in terms of u and again setting $\theta=\pi/2$ to obtain

$$\begin{aligned}\frac{2a}{\pi}\int_{\phi_1}^{\phi_2}\frac{f(\phi)}{\epsilon}G_n d\phi&=\frac{2a\beta}{\pi\epsilon}\int_{-1}^{1}\frac{F(u)G_n}{\sin\phi}du\\&=\frac{2a\beta}{\pi\epsilon}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{C_mA_{n+1}\cos\left(\frac{n\pi}{2}\right)}{(n+1)\pi}\\&\quad\cdot\int_{-1}^{1}\frac{T_m(u)\sin(n+1)\phi}{\left[1-u^2\right]^{1/2}\sin\phi}du.\end{aligned} \tag{B2}$$

The last integration in (B2) can be evaluated by substituting the following expansion:

$$\frac{\sin(n+1)\phi}{\sin\phi}=\sum_{k=0}^{n}p_{kn}T_k(u) \tag{B3}$$

and then using the orthogonality relationship for the Chebyshev polynomials:

$$\int_{-1}^{1}\frac{T_m(u)T_k(u)}{\left[1-u^2\right]^{1/2}}du=\frac{\pi}{2\Delta_m}\delta_{km} \tag{B4}$$

where Δ_m is the Neumann factor defined as

$$\Delta_m=\begin{cases}1/2, & m=0\\ 1, & m>0\end{cases}$$

and δ_{km} is the Kronecker delta defined as

$$\delta_{km}=\begin{cases}1, & k=m\\ 0, & k\neq m\end{cases}.$$

Thus

$$\frac{2a}{\pi}\int_{\phi_1}^{\phi_2}\frac{f(\phi)}{\epsilon}G_n d\phi=\frac{a\beta}{\pi\epsilon}\sum_{n=0}^{\infty}\sum_{k=0}^{n}p_{kn}\frac{C_k}{\Delta_k}A_{n+1}\frac{\cos\left(\frac{n\pi}{2}\right)}{(n+1)}$$

and (B1) reduces to

$$\sum_{n=0}^{\infty}C_nJ_n+2\sum_{n=0}^{\infty}\sum_{k=0}^{n}p_{kn}\frac{C_k}{\Delta_k}A_{n+1}\frac{\cos\left(\frac{n\pi}{2}\right)}{(n+1)}=\frac{2\pi\epsilon V}{a\beta}.$$

The remaining set of equations is found by substituting (7) into (10) and using the following expansion:

$$\cos m\theta=\sum_{k=0}^{m}q_{km}U_k(v)$$

to obtain

$$H(v)=\frac{-2}{\pi}\sum_{m=1}^{\infty}\sum_{k=0}^{m}A_mR_mq_{km}U_k(v) \tag{B5}$$

where

$$R_m\equiv\int_{\phi_1}^{\phi_2}f(\phi)\,\sin m\phi\,d\phi. \tag{B6}$$

When (B5) is equated to (11) and coefficients of $U_m(v)$ are matched, the following equation results:

$$C_{n+1} = \frac{-2}{\pi} \sum_{m=n}^{\infty} A_m R_m q_{nm}, \qquad n=0,1,2,\cdots. \quad \text{(B7)}$$

R_m in (B7) can be evaluated by transforming (B6) into an equation in terms of u, substituting for $F(u)$ from (13), and using the expansion given in (B3) and the orthogonality relationship given in (B4). The resulting expression is given as

$$R_m = \frac{\pi\beta}{2}(1-\delta_{m0}) \sum_{k=0}^{m-1} \frac{C_k}{\Delta_k} p_{k,m-1}.$$

Thus (B7) reduces to

$$C_{n+1} = -\beta \sum_{m=n}^{\infty} \sum_{k=0}^{m-1} (1-\delta_{m0}) A_m q_{nm} \frac{C_k}{\Delta_k} p_{k,m-1}, \qquad n=0,1,2,\cdots.$$

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The Compensation of Step Discontinuities in TEM-Mode Transmission Lines

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***Abstract*—A method for the compensation of the effects due to the discontinuities that arise when transmission lines of different characteristic impedance are joined is presented. The proposed method is not based on calculating the equivalent circuit of the discontinuity but makes use of a simple taper on the wider line at an impedance step to remove the effects of the discontinuity.**

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I. INTRODUCTION

THE EFFECTS observed when lines of different characteristic impedance are joined to form a step discontinuity are well known, and various authors have presented equivalent circuits [1], [2]. The parameters of such equivalent circuits have to be incorporated into the design, which can lead to considerable complication. In the limiting case, the end effect observed in open-circuit stubs or open-circuit parallel-coupled lines can be regarded as a step from finite to zero linewidth. In the latter case,